

Landscape disruption effects in a meta-epidemic model with steady state demographics and migrations saturation.

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Abstract

We continue the investigations of an ecosystem where a epidemic-affected population can move between two connected patches, [1], by considering what happens to the system when the migration paths are interrupted in one direction, or when the infected are not able to exert the effort for migrating into the other patch.

1 Introduction

In [1] an epidemic-affected one species metapopulation model with fixed size and immigrations depending inversely on the crowding of the arrival environment has been introduced, along lines that allow disease consideration in fragmented habitats, [3]. Here, we add reproduction capabilities and specialize the system to two particular cases, when the migrations can occur only in one direction, or when infected are too weak to undertake any migrating effort.

2 Unidirectional migrations

Assume it is not possible to return to patch 1 from the second one. The system is pictured in Figure 1 left. The model reads

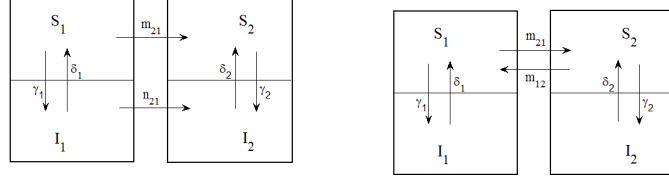


Figure 1: Left: no migrations from patch 2 into patch 1. Right: Infected do not migrate.

$$\dot{S}_1 = r_1 S_1 - \gamma_1 S_1 I_1 + \delta_1 I_1 - m_{21} \frac{S_1}{A + I_2 + S_2}, \quad \dot{I}_1 = \gamma_1 S_1 I_1 - (\delta_1 + \mu_1) I_1 - n_{21} \frac{I_1}{B + I_2 + S_2}, \quad (1)$$

$$\dot{S}_2 = r_2 S_2 - \gamma_2 S_2 I_2 + \delta_2 I_2 + m_{21} \frac{S_1}{A + I_2 + S_2}, \quad \dot{I}_2 = \gamma_2 S_2 I_2 - (\delta_2 + \mu_2) I_2 + n_{21} \frac{I_1}{B + I_2 + S_2}, \quad (2)$$

where r_k , $k = 1, 2$ represent the net reproduction rates of the population in each environment, which is assumed to have different ecological characteristics. The other parameters have the following meanings μ_k is the infected mortality rate in each patch, γ_k the disease contact rate, δ_k is the disease recovery rate, A is the half saturation constant for the susceptibles, and B the one for the infected; finally the migration rates from patch j into patch i are m_{ij} for the susceptibles and n_{ij} for the infected.

The equilibria are the origin, trivially, possibly the coexistence in both patches with an endemic disease, and the point with only the arrival patch populated by both susceptibles and infected, $X_1 = (0, 0, \tilde{S}_2, \tilde{I}_2)$,

$$\tilde{S}_2 = \frac{\delta_2 + \mu_2}{\gamma_2}, \quad \tilde{I}_2 = \frac{r_2(\delta_2 + \mu_2)}{\gamma_2 \mu_2},$$

which is clearly unconditionally feasible, and the point $X_2 = (\tilde{S}_1, 0, \tilde{S}_2, \tilde{I}_2)$, with the first patch disease-free,

$$\tilde{S}_2 = \frac{\delta_2 + \mu_2}{\gamma_2}, \quad \tilde{I}_2 = \frac{\gamma_2 m_{21} - r_1 \gamma_2 A - (\delta_2 + \mu_2) r_1}{r_1 \gamma_2},$$

$$\tilde{S}_1 = \frac{-(\delta_2 + \mu_2) r_1 r_2 + \mu_2 (\gamma_2 m_{21} - r_1 \gamma_2 A - (\delta_2 + \mu_2) r_1)}{r_1 \gamma_2}.$$

This equilibrium is feasible for

$$\gamma_2 m_{21} \geq r_1 \gamma_2 A + (\delta_2 + \mu_2) r_1, \quad \mu_2 [\gamma_2 m_{21} - r_1 \gamma_2 A - (\delta_2 + \mu_2) r_1] \geq (\delta_2 + \mu_2) r_1 r_2. \quad (3)$$

For the equilibrium with both patches populated and with endemic disease, let us sum the first and third equations of (1) as well as the second and

fourth one, to obtain

$$r_1\tilde{S}_1 - \gamma_1\tilde{S}_1\tilde{I}_1 + \delta_1\tilde{I}_1 + r_2\tilde{S}_2 - \gamma_2\tilde{S}_2\tilde{I}_2 + \delta_2\tilde{I}_2 = 0, \quad \gamma_1\tilde{S}_1\tilde{I}_1 - (\delta_1 + \mu_1)\tilde{I}_1 + \gamma_2\tilde{S}_2\tilde{I}_2 - (\delta_2 + \mu_2)\tilde{I}_2 = 0. \quad (4)$$

Adding these equations further and solving for \tilde{S}_1 as function of $\tilde{I}_1, \tilde{S}_2, \tilde{I}_2$ we substitute it into the second one of (4) to get

$$\tilde{S}_1 = \frac{-r_2\tilde{S}_2 + \mu_1\tilde{I}_1 + \mu_2\tilde{I}_2}{r_1}, \quad \tilde{S}_2 = \frac{r_1((\delta_1 + \mu_1)\tilde{I}_1 + (\delta_2 + \mu_2)\tilde{I}_2) - \gamma_1\mu_1\tilde{I}_1^2 - \gamma_1\mu_1\tilde{I}_1\tilde{I}_2}{r_1\gamma_2\tilde{I}_2 - r_2\gamma_1\tilde{I}_1}.$$

Necessary conditions for the feasibility of this equilibrium are either one of the following two sets of inequalities

$$\tilde{I}_1 > \frac{r_1\gamma_2\tilde{I}_2}{r_2\gamma_1}, \quad \tilde{S}_2 < \frac{\mu_1\tilde{I}_1 + \mu_2\tilde{I}_2}{r_2}, \quad \tilde{I}_2 > \frac{\gamma_1\mu_1\tilde{I}_1^2 - r_1(\delta_1 + \mu_1)\tilde{I}_1}{r_1(\delta_2 + \mu_2) - \gamma_1\mu_2\tilde{I}_1} \equiv Z; \quad (5)$$

$$\tilde{I}_1 < \frac{r_1\gamma_2\tilde{I}_2}{r_2\gamma_1}, \quad \tilde{S}_2 > \frac{\mu_1\tilde{I}_1 + \mu_2\tilde{I}_2}{r_2}, \quad \tilde{I}_2 < Z. \quad (6)$$

But we need also to ensure that $Z > 0$, so that finally we also get either one of the inequalities

$$\frac{\delta_1 + \mu_1}{\gamma_1\mu_1} < I_1 < \frac{\delta_2 + \mu_2}{\gamma_1\mu_2}; \quad \frac{\delta_2 + \mu_2}{\gamma_1\mu_2} < I_1 < \frac{\delta_1 + \mu_1}{\gamma_1\mu_1}. \quad (7)$$

The Jacobian of (1) is

$$J = \begin{bmatrix} -\gamma_1 I_1 - \eta_1 + r_1 & -\gamma_1 S_1 + \delta_1 & \eta_2 S_1 & \eta_2 S_1 \\ \gamma_1 I_1 & \gamma_1 S_1 - \delta_1 - \mu_1 - \theta_1 & \theta_2 I_1 & \theta_2 I_1 \\ \eta_1 & 0 & -\gamma_2 I_2 - \eta_2 S_1 + r_2 & -\gamma_2 S_2 + \delta_2 - \eta_2 S_1 \\ 0 & \theta_1 & \gamma_2 I_2 - \theta_2 I_1 & \gamma_2 S_2 - \delta_2 - \mu_2 - \theta_2 I_1 \end{bmatrix}$$

where

$$\eta_1 = \frac{m_{21}}{A + S_2 + I_2}, \quad \eta_2 = \frac{m_{21}}{(A + S_2 + I_2)^2}, \quad \theta_1 = \frac{n_{21}}{B + S_2 + I_2}, \quad \theta_2 = \frac{n_{21}}{(B + S_2 + I_2)^2}.$$

The origin is unstable, since the eigenvalues are $r_2, -\delta_2 - \mu_2, (r_1 A - m_{21})A^{-1}, -(\delta_1 B + n_{21} + \mu_1 B)B^{-1}$.

At X_1 we have instead one rather complicated but negative eigenvalue, $\lambda_1 < 0$ and

$$\lambda_2 = \frac{-\mu_2 m_{21} \gamma_2 + \mu_2 r_1 A \gamma_2 + \mu_2 r_1 \delta_2 + r_1 \mu_2^2 + r_1 r_2 (\delta_2 + \mu_2)}{\mu_2 (A \gamma_2 + \delta_2 + \mu_2 \delta_2 + r_2)},$$

$$\lambda_{3,4} = \frac{-r_2\delta_2 \pm \sqrt{r_2^2\delta_2^2 - 4\mu_2^2r_2(\mu_2 + \delta_2)}}{2\mu_2} < 0.$$

Stability is then obtained for

$$\mu_2r_1A\gamma_2 + \mu_2r_1\delta_2 + r_1\mu_2^2 + r_1r_2(\delta_2 + \mu_2) < \mu_2m_{21}\gamma_2. \quad (8)$$

At X_2 , one eigenvalue is explicit, $\lambda_1 = \gamma_1\tilde{S}_1 - \delta_1 - \mu_1 - n_{21}[B + \tilde{S}_2 + \tilde{I}_2]^{-1}$ while the remaining ones are the roots of the cubic equation $\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0$, with $a_2 = \gamma_2\tilde{I}_2 + m_{21}\tilde{S}_1D^{-2} - r_2 - \gamma_2\tilde{S}_2 - \delta_2 + \mu_2$ and

$$\begin{aligned} a_1 &= \left(\frac{m_{21}}{D} - r_1\right), \quad D = A + \tilde{S}_2 + \tilde{I}_2 \\ a_2 &= \gamma_2\mu_2\tilde{I}_2 + r_2(\gamma_2\tilde{S}_2 - \delta_2 - \mu_2) + \frac{m_{21}\tilde{S}_1}{D^2} \left(-\gamma_2\tilde{S}_2 + \delta_2 + \mu_2 + \gamma_2\tilde{I}_2 + \frac{m_{21}}{D}\right), \\ a_0 &= \left(\frac{m_{21}}{D} - r_1\right) \left[\gamma_2\tilde{I}_2 \left(\mu_2 + \frac{m_{21}\tilde{S}_1}{D^2}\right) + \left(r_2 - \frac{m_{21}\tilde{S}_1}{D^2}\right) (\gamma_2\tilde{S}_2 - \delta_2 - \mu_2) \right] \\ &\quad + \frac{m_{21}^2}{D^3} \tilde{S}_1 (\gamma_2\tilde{S}_2 - \gamma_2\tilde{I}_2 - \delta_2 - \mu_2). \end{aligned}$$

The Routh-Hurwitz conditions and negativity of the first eigenvalue guarantee stability for

$$\lambda_1 < 0, \quad a_0 > 0, \quad a_2 > 0, \quad a_2a_1 > a_0. \quad (9)$$

A Hopf bifurcation would be possible if $a_2a_1 = a_0$.

The points X_1 , X_2 and coexistence can stably be achieved respectively by the following parameter choices

$$\begin{aligned} r_1 &= 2, \quad r_2 = 1, \quad \gamma_1 = 0.5, \quad \gamma_2 = 1, \quad \delta_1 = 0.5, \quad \delta_2 = 2, \quad \mu_1 = \mu_2 = 1, \\ &\quad m_{21} = 20, \quad n_{21} = 0.5, \quad A = B = 1; \\ r_1 &= r_2 = \gamma_1 = 1, \quad \gamma_2 = 0.5, \quad \delta_1 = 1, \delta_2 = \mu_1 = 1, \quad \mu_2 = 3, \\ &\quad m_{21} = 30, \quad n_{21} = A = B = 1; \\ r_1 &= r_2 = 1, \quad \gamma_1 = 0.5, \quad \gamma_2 = \delta_1 = 1, \quad \delta_2 = 2, \quad \mu_1 = 1, \quad \mu_2 = 2, \\ &\quad m_{21} = 1, \quad n_{21} = 0.5, \quad A = 1, \quad B = 3. \end{aligned}$$

3 No Infected Migrations

In this case the model with $n_{12} = n_{21} = 0$ is shown in Figure 1 right and reads

$$\begin{aligned}\dot{S}_1 &= r_1 S_1 - \gamma_1 S_1 I_1 + \delta_1 I_1 - m_{21} \frac{S_1}{A + I_2 + S_2} + m_{12} \frac{S_2}{A + S_1 + I_1}, \\ \dot{I}_1 &= \gamma_1 S_1 I_1 - (\delta_1 + \mu_1) I_1, \\ \dot{S}_2 &= r_2 S_2 - \gamma_2 S_2 I_2 + \delta_2 I_2 + m_{21} \frac{S_1}{A + I_2 + S_2} - m_{12} \frac{S_2}{A + S_1 + I_1}, \\ \dot{I}_2 &= \gamma_2 S_2 I_2 - (\delta_2 + \mu_2) I_2.\end{aligned}\quad (10)$$

In addition to the origin and ecosystem survival with the endemic disease in both patches, we find two more points

$$U = \left(\frac{\delta_1 + \mu_1}{\gamma_1}, r_1 \frac{\delta_1 + \mu_1}{\gamma_1 \mu_1} + \frac{r_2 \tilde{S}_2}{\mu_1}, \tilde{S}_2^U, 0 \right), \quad W = \left(\tilde{S}_1^W, 0, \frac{\delta_2 + \mu_2}{\gamma_2} r_2 \frac{\delta_2 + \mu_2}{\gamma_2 \mu_2} + \frac{r_1 \tilde{S}_1}{\mu_2} \right),$$

where \tilde{S}_2^U and \tilde{S}_1^W solve the equations

$$\begin{aligned}(r_2 \tilde{S}_2 A + r_2 \tilde{S}_2^2 + m_{21} \tilde{S}_1) \tilde{I}_1 &= m_{12} \tilde{S}_2 (A + \tilde{S}_2) \\ -r_2 \tilde{S}_2 (A^2 + A \tilde{S}_1 + A \tilde{S}_2 + \tilde{S}_1 \tilde{S}_2) - m_{21} \tilde{S}_1 (A + \tilde{S}_1),\end{aligned}\quad (11)$$

$$\begin{aligned}(r_1 \tilde{S}_1 A + r_1 \tilde{S}_1^2 + m_{12} \tilde{S}_2) \tilde{I}_1 &= m_{21} \tilde{S}_1 (A + \tilde{S}_1) \\ -r_1 \tilde{S}_1 (A^2 + A \tilde{S}_1 + A \tilde{S}_2 + \tilde{S}_1 \tilde{S}_2) - m_{12} \tilde{S}_2 (A + \tilde{S}_2).\end{aligned}\quad (12)$$

The first is an intersection problem of the curves $g(\tilde{S}_2) = a_2 \tilde{S}_2^2 + a_1 \tilde{S}_2 + a_0$, $f(\tilde{S}_2) = b_3 \tilde{S}_2^3 + b_2 \tilde{S}_2^2 + b_1 \tilde{S}_2 + b_0$, with

$$a_2 = m_{12} - r_2 A - r_2 \frac{\delta_1 + \mu_1}{\gamma_1}, \quad a_1 = A a_2, \quad (13)$$

$$a_0 = -m_{21} A \frac{\delta_1 + \mu_1}{\gamma_1} - m_{21} \frac{(\delta_1 + \mu_1)^2}{\gamma_1^2} < 0$$

$$b_3 = \frac{r_2^2}{\mu_1}, \quad b_2 = r_1 r_2 \frac{\delta_1 + \mu_1}{\gamma_1 \mu_1} + \frac{r_2^2 A}{\mu_1}, \quad (14)$$

$$b_1 = r_1 r_2 A \frac{\delta_1 + \mu_1}{\gamma_1 \mu_1} + r_2 m_{21} \frac{\delta_1 + \mu_1}{\gamma_1 \mu_1}, \quad b_0 = r_1 m_{21} \frac{(\delta_1 + \mu_1)^2}{\gamma_1^2 \mu_1}.$$

The parabola g has roots $\tilde{S}_2^\pm = -[a_1 \gamma_1^2 A \pm \sqrt{a_1^2 \gamma_1^4 A^2 + a_2 K}] (2a_2 \gamma_1)^{-2}$ and an intersection for $\tilde{S}_2 \geq 0$ is possible only for $a_2 > 0$. But the existence of the intersection is not ensured, since the cubic has positive coefficient, $b_i \geq 0$,

$i = 0, \dots, 3$. Similar remarks hold for the point W . Necessary conditions for feasibility are respectively

$$\tilde{S}_2 > \tilde{S}_2^- = -\frac{a_1\gamma_1^2 - \sqrt{a_1^2\gamma_1^4 + a_2K}}{2a_2\gamma_1^2}, \quad \tilde{S}_1 > \tilde{S}_1^- = -\frac{c_1\gamma_2A - \sqrt{c_1^2\gamma_2^2A^2 + c_2H}}{2c_2\gamma_2}.$$

For feasibility of the coexistence equilibrium the necessary conditions (5), (6) and (7) still hold.

The Jacobian of (10) is

$$J = \begin{bmatrix} -\gamma_1 I_1 - \alpha_1 - \beta_2 S_2 + r_1 & -\gamma_1 S_1 + \delta_1 - \beta_2 S_2 & \alpha_2 S_1 + \beta_1 & \alpha_2 S_1 \\ \gamma_1 I_1 & \gamma_1 S_1 - \delta_1 - \mu_1 & 0 & 0 \\ \alpha_1 + \beta_2 S_2 & \beta_2 S_2 & -\gamma_2 I_2 - \alpha_2 S_1 + \beta_1 + r_2 & -\gamma_2 S_2 + \delta_2 - \alpha_2 S_1 \\ 0 & 0 & \gamma_2 I_2 & \gamma_2 S_2 - \delta_2 - \mu_2 \end{bmatrix}$$

where

$$\alpha_1 = \frac{m_{21}}{A + I_2 + S_2}, \quad \alpha_2 = \frac{m_{21}}{(A + I_2 + S_2)^2}, \quad \beta_1 = \frac{m_{12}}{A + I_1 + S_1}, \quad \beta_2 = \frac{m_{12}}{(A + I_1 + S_1)^2}.$$

The origin is unstable as the eigenvalues are $-\delta_1 - \mu_1 < 0$, $-\delta_2 - \mu_2 < 0$, $[k \pm \sqrt{k^2 + 4A(r_1 m_{12} + r_2 m_{21} r_2 A)}]A^{-1}$, for which one is positive independently of the sign of $k = r_1 A + r_2 A - m_{12} - m_{21}$.

At U , one eigenvalue is explicit, the other ones are the roots of the cubic $\sum_{k=0}^3 \lambda^k p_k = 0$, for which the Routh-Hurwitz conditions give stability for

$$p_0 > 0, \quad p_2 > 0, \quad p_2 p_1 > p_0, \quad \gamma_2 \tilde{S}_2 < \delta_2 + \mu_2 \quad (15)$$

where

$$\begin{aligned} p_2 &= \gamma_1 \tilde{I}_1 + \hat{\alpha}_1 + \beta_2 - r_1 - \gamma_1 \tilde{S}_1 + \delta_1 + \mu_1 + \hat{\alpha}_2 - \beta_1 - r_2, \\ p_1 &= -(\hat{\alpha}_2 + \beta_1)(\hat{\alpha}_1 + \beta_2) - \gamma_1 \tilde{I}_1(\delta_1 - \gamma_1 \tilde{S}_1 - \beta_2) \\ &\quad + (r_1 - \gamma_1 \tilde{I}_1 - \hat{\alpha}_1 - \beta_2)(\gamma_1 \tilde{S}_1 - \delta_1 - \mu_1) \\ &\quad + (r_1 - \gamma_1 \tilde{I}_1 - \hat{\alpha}_1 - \beta_2)(r_2 - \hat{\alpha}_2 + \beta_1) + (\gamma_1 \tilde{S}_1 - \delta_1 - \mu_1)(r_2 - \hat{\alpha}_2 + \beta_1), \\ p_0 &= (r_2 - \hat{\alpha}_2 + \beta_1)(+\gamma_1 \tilde{I}_1(\delta_1 - \gamma_1 \tilde{S}_1 - \beta_2) \\ &\quad - (\hat{\alpha}_2 + \beta_1)(\gamma_1 \beta_2 \tilde{I}_1 - (\gamma_1 \tilde{S}_1 - \delta_1 - \mu_1)(\hat{\alpha}_1 + \beta_2)) \\ &\quad - (r_1 - \gamma_1 \tilde{I}_1 - \hat{\alpha}_1 - \beta_2)(\gamma_1 \tilde{S}_1 - \delta_1 - \mu_1)) \end{aligned}$$

A similar situation occurs for W , giving stability for $\gamma_1 \tilde{S}_1 < \delta_1 + \mu_1$ in place of the last one (15), and the same other conditions with q_i in place of p_i ,

where

$$\begin{aligned}
q_2 &= \alpha_1 + \hat{\beta}_2 - r_1 + \gamma_2 \tilde{I}_2 - \alpha_2 - \hat{\beta}_1 - r_2 - \gamma_2 \tilde{S}_2 + \delta_2 + \mu_2, \\
q_1 &= (\gamma_2 \tilde{S}_2 - \delta_2 - \mu_2)(r_2 - \alpha_1 - \hat{\beta}_2 + r_1 - \gamma_2 \tilde{I}_2 + \alpha_2 + \hat{\beta}_1) \\
&\quad + (r_1 - \alpha_1 - \hat{\beta}_2)(r_2 - \gamma_2 \tilde{I}_2 + \alpha_2 + \hat{\beta}_1) \\
&\quad - (\alpha_1 + \hat{\beta}_2)(\alpha_2 + \hat{\beta}_1) - \gamma_2 \tilde{I}_2(\delta_2 - \gamma_2 \tilde{S}_2 - \alpha_2), \quad q_0 = (\alpha_2 + \hat{\beta}_1)(\alpha_1 + \hat{\beta}_2) \\
&\quad + (\gamma_2 \tilde{S}_2 - \delta_2 - \mu_2)((\alpha_1 + \hat{\beta}_2 - r_1)(-\gamma_2 \tilde{I}_2 \\
&\quad + \alpha_2 + \hat{\beta}_1 + r_2) + \gamma_2 \tilde{I}_2((- \alpha_1 - \hat{\beta}_2 + r_1)(-\gamma_2 \tilde{S}_2 + \delta_2 - \alpha_2) - \alpha_2(\alpha_1 + \hat{\beta}_2))).
\end{aligned}$$

Numerical simulations reveal that U , W and coexistence can be stably achieved, respectively for the values

$$\begin{aligned}
r_1 &= 1, \quad r_2 = 0.5, \quad \gamma_1 = 0.5, \quad \gamma_2 = \delta_1 = 1, \quad \delta_2 = 0.2, \quad \mu_1 = 1, \\
&\quad \mu_2 = m_{21} = 10, \quad m_{12} = 17, \quad A = 10; \\
r_1 &= 0.2, \quad r_2 = \gamma_1 = 1, \quad \gamma_2 = 2, \quad \delta_1 = 1.8, \quad \delta_2 = 0.3, \quad \mu_1 = 1, \\
&\quad \mu_2 = 6, \quad m_{21} = 8.8, \quad m_{12} = 4, \quad A = 10; \\
r_1 &= r_2 = \gamma_1 = \gamma_2 = 1, \quad \delta_1 = 2, \quad \delta_2 = 0.5, \quad \mu_1 = 2\mu_2 = 2, \\
&\quad m_{12} = 3, \quad m_{21} = 1, \quad A = 10.
\end{aligned}$$

4 Interpretation

The generic equilibria share common properties in both models. Namely, instability of the origin means ecosystem permanence. The systems allow also survival of the population in both patches, with endemic disease.

Within the unidirectional migration model, the patch from which migrations occur could become completely depleted, or else it may be populated, but disease-free. As these are mutually exclusive equilibria, it is possible that bistability phenomena arise as for [1]. The basins of attraction of the equilibria could be determined using the algorithms being developed, [2].

For the model in which infected do not migrate, specific equilibria are the situations in which either patch becomes epidemic-free. These considerations could be very useful in practical situations for disease eradication in some environments.

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